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Umbrella-Type Surfaces in Architecture of Spatial Structures

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Abstract : This paper provides information on the 16 new umbrella-type surfaces and on two umbrella surfaces from the identical parts of translational surface, formed by a generating circle of constant radius, using methods of differential geometry and computer design. Parametrical equations of the surfaces are presented and the influence of constant parameters containing in the equations of these surfaces on their form is studied in detail. All surfaces are pictured by means of computer graphics. Architecture of spatial structures can be a main sphere of their application.

Keywords: *architectural engineering, differential geometry, geometric design, spatial structures, translational surface, umbrella-type surface, umbrella surface*

I. INTRODUCTION

The centuries-old experience of building of stone domes gave rise to domes of various forms. The domes with ribs known in architecture as umbrella shells attract special attention. Besides strength, rigidity, and stability, they possess aesthetic and economic advantages and modern thin-walled structures give an opportunity to add lightness and airiness to them. The Roman emperor Hadrian first applied an umbrella shell as a covering of the villa in Tivoli built in 127–134 years. Until the middle of the 20th century, any principles of design of middle surfaces of umbrella domes were absent, and it delayed the development of methods of strength analysis. The erection of a reinforced concrete structure of an umbrella type without intermediate supports in Royan (France) is a new progressive phenomenon in the history of the application of umbrella shells. Horizontal sections of this covering are sinusoidal curves but diagonals have parabolic form. P. L. Nervi and F. Candela also worked with umbrella shells and designed interesting buildings, for example, Church of the Priority, St. Louis, USA and Oceanographic, Valencia, Spain. Until present time, umbrella and umbrella-type shells were built in many countries [1], but the author didn't find equations defining middle surfaces of these shells, and it hampers the application of computer-designed shell buildings.

II. GENERAL INFORMATION

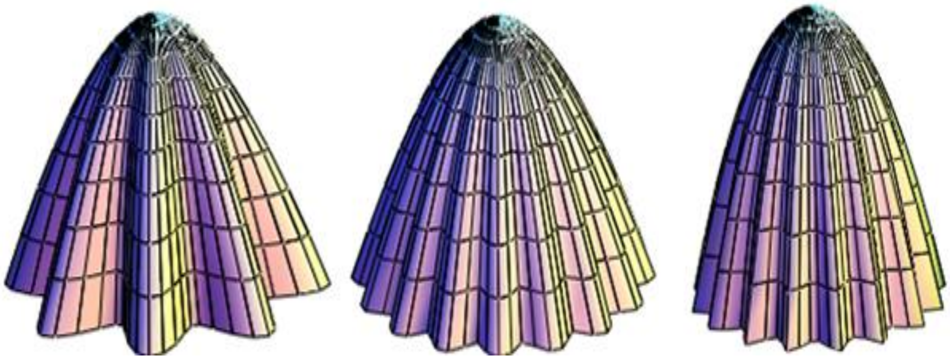
In the Dictionary of Architectural and Building Technology [2], one can read that an umbrella shell is a shell roof formed by four hypar shells, or by other suitable arrangements of hypar shells. But this definition narrows the class of umbrella surfaces. It is better to name these surfaces as hyperbolic paraboloid umbrella surfaces [3] or inverted, doubly-curved umbrella, hyperbolic paraboloid surfaces [4]. According to another definition, an umbrella dome is a cyclical symmetrical spatial structure formed by several identical elements, but curves derived as a result of intersection of the middle surfaces of identical elements are generatrixes of any surface of revolution. In some papers, umbrella domes are called multiple, pumpkin, melon, scalloped, or parachute domes. Contour surface is called dome-formed surface of revolution on which contour curves of dome elements are laid. Contour curves of the element are the curves limiting the contour of a middle surface of the dome element. Surfaces of umbrella type or umbrella-type surfaces are called cyclical symmetrical surfaces consisting of several identical elements. But a complete surface of umbrella type and all surfaces of its identical elements are described by the same Cartesian equation or by the same parametrical equations.

A monograph of G. Brankov [5] may be one of the first books on geometry of surfaces of umbrella type. Later, the descriptions of umbrella surfaces and surfaces of umbrella type as applied to realizable factory-made goods or buildings began to be published [6]. In this paper, a class of surfaces of umbrella type is supplemented and broadened, and it can result in expansion of the sphere of their application. Some of the umbrella forms were already applied, others are in the initial stage of their application, and the third will be used in the future, while the remaining ones will have only purely theoretical meaning. In this paper, the 16 new umbrella-type surfaces are proposed for the introduction in the architecture. Two new umbrella surfaces were applied in the real computer-aided design of the entertainment center. All equations of the examined surfaces were derived by the author with carrying out of the condition that a plane meridian passes through any point of the foot-directing curve and the peak of an umbrella-type surface with the coordinates (0, 0, h). The analogous approach will be used in all parts of the paper. Availability of parametrical equations of umbrella-type surfaces clears wide possibilities for computer design.

III. UMBRELLA-TYPE SURFACES WITH PARABOLIC MERIDIANS

A crimped paraboloid of revolution (Fig. 1a) has a circular sinusoid

$$x = (R + a\cos n\varphi)\cos\varphi,\ y = (R + a\cos n\varphi)\sin\varphi,\ z = 0 \quad (1)$$



(a) (b) (c)

Fig. 1. Umbrella-type surfaces with parabolic meridians: (a) a crimped paraboloid of revolution; (b) a crimped paraboloid of revolution with external crimps; (c) a crimped paraboloid of revolution with inner crimps as the foot. Equation (1) contains n = number of peaks of a sinusoid on the circular plan, a = amplitude of waves on the foot of a surface, R = radius of a foot circle of the paraboloid, relative to which a circular sinusoid was formed. The parametrical equations of a crimped paraboloid of revolution (Fig. 1a) can be written as

$$x = x(r,\varphi) = r\left(1 + \frac{ar\cos n\varphi}{R^2}\right)\cos\varphi,$$

$$y = y(r,\varphi) = r\left(1 + \frac{ar\cos n\varphi}{R^2}\right)\sin\varphi,$$

$$z = z(r) = h\left(1 - \frac{r^2}{R^2}\right),$$

where $0 \le z \le h$; $0 \le \varphi \le 2\pi$; $0 \le r \le R$, and h is a height of a crimped paraboloid of revolution. Graphical description of a surface of crimped paraboloid of revolution and the name of this surface was presented first in a monograph [5].

A crimped paraboloid of revolution with external crimps (Fig. 1(b)) has a circular wavy curve in the foot

$$x = (R + \delta a|\cos n\varphi|)\cos\varphi,\ y = (R + \delta a|\cos n\varphi|)\sin\varphi,\ z = 0 \quad (2)$$

with picks directed from the center of the foot, $\delta = 1$.

This surface can be given by the following parametrical equations:

$$x = x(r,\varphi) = r\left(1 + \delta\frac{ar|\cos n\varphi|}{R^2}\right)\cos\varphi,$$

$$y = y(r,\varphi) = r\left(1 + \delta\frac{ar|\cos n\varphi|}{R^2}\right)\sin\varphi,$$

$$z = z(r) = h\left(1 - \frac{r^2}{R^2}\right), \quad (3)$$

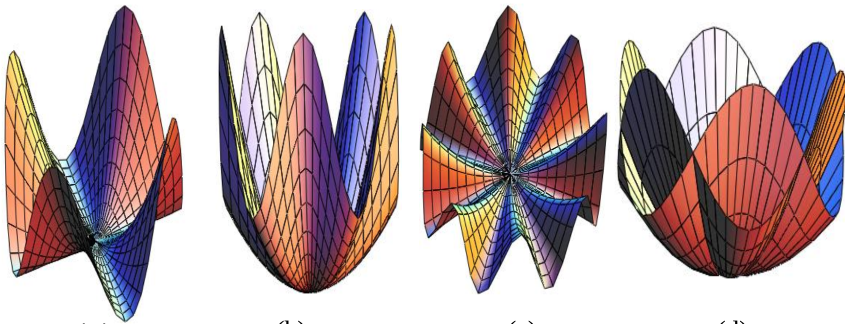
where $0 \le z \le h$; $0 \le \varphi \le 2\pi$; $0 \le r \le R$, $\delta = 1$.

A crimped paraboloid of revolution with inner crimps (Fig. 1c) has a circular wavy curve (2) with $\delta = -1$ in the foot with picks directed only inside the circular foot. This surface may be defined by the parametrical equations (3) where $\delta = -1$.

A paraboloid of revolution with radial waves is formed by plane parabolas the picks of which coincide with a central fixed point. Tangents drawn through the central points of parabolas must be in the same plane. Any cross section of the surface by the plane passing through the Oz axis will be a parabola.
The parametrical equations of this surface of umbrella type can be represented as

$$x = x(u, v) = u \cos v,\ y = y(u, v) = u \sin v,\ z = z(u, v) = [a \sin(nv) + b]\, u^2, \qquad (4)$$

where v is an angle taken from the Ox axis in the direction of the Oy axis; a = const is an amplitude of the wave; n is a number of peaks of the waves; and b is a constant parameter of the base paraboloid of evolution. The curvilinear coordinate lines u, v are not lines of principle curvatures.



(a) $(0 \le u \le 1)$ $a = b = 1; n = 4$ (b) $(0 \le u \le 1)$ $a = b = 1; n = 4$ (c) $(0 \le u \le 1)$ $a = b = 1; n = 4$ (d) $(-1 \le u \le 1)$ $a = 0{,}5; b = 1; n = 3$

Fig. 2. A paraboloid of revolution with radial waves: (a) a = b; (b) b > a; (c) b < a; (d) n = 3, v > 2π

If we take a = b then lower wave picks will be on the xOy plane (Fig. 2a). If b > a then a surface (4) will be in the area of positive values of the ordinate z (Fig. 2b). If b < a then a surface (4) will have both positive and negative values of the ordinate z (Fig. 2c). A paraboloid of revolution with radial waves will degenerate into a paraboloid of revolution if a = 0.

If n is an even number then a surface (4) will not have lines of self-intersection of the surface when v > 2π. If n is an odd number and $0 \le u \le u_0$ then self-intersections of a surface (4) will not be, but the combs will appear if $-u_0 \le u \le u_0$.

The surface with n = 3, 0 ≤ v ≤ 2π; –1 ≤ u ≤ 1 is presented in Fig. 2d. A surface (4) will be a comb surface if we take fractional value of a parameter n and v > 2π.

Surfaces of umbrella-type with parabolic generatrixes and a circle on one contour are interesting from a geometrical point of view. For example, the surfaces shown in Figs. 3a and 3b may be defined by the following parametrical equations:

$$x = x(z,\varphi) = \{a - [a - v(\varphi)]z^2 / h^2\} x(\varphi)/v(\varphi);$$
$$y = y(z,\varphi) = \{a - [a - v(\varphi)]z^2 / h^2\} y(\varphi)/v(\varphi);\ z = z,$$

where $v(\varphi) = \sqrt{R^2 + p(1 - \cos n\varphi)}$. The surface examined is formed by single-parametric family of parabolas lying in planes of a pencil passing through the coordinate axis Oz. It should be noted that the axes of parabolas are in the horizontal plane z = 0 but picks of the parabolas are placed on the circle with a radius a and with a center in the point O (0; 0; 0). A surface has an epicycloid in the cross-section z = h (Fig. 3a).

$$x = x(\varphi) = (R + r) \cos \varphi - r \cos (1 + n) \varphi,\ y = y(\varphi) = (R + r) \sin \varphi - r \sin (1 + n) \varphi, \qquad (5)$$

where n is a number of external peaks of the epicycloid; n = R/r; R is a radius of the circle on which another circle with radius r rolls on the outside and any point of which forms an epicycloid; φ is an angle taken from the Ox axis in the direction of the Oy axis or a hypocycloid (Fig. 3b)

(a) R = lm; n = 5; a = 3m; h = 4,5m (b) R = lm; n = 5; a = 3m; h = 4,5m

Fig. 3. Umbrella-type surfaces with parabolic meridians: (a) a surface with an epicycloid in the cross-section z = h; (b) a surface with a hypocycloid in the cross-section z = h

$$x = x(\varphi) = (R - r)\cos\varphi + r\cos(1 - n)\varphi,\quad y = y(\varphi) = (R - r)\sin\varphi - r\sin(1 - n)\varphi, \tag{6}$$

where n is a number of peaks of the hypocycloid; R is a radius of the circle on which another circle with radius r rolls on the inside and any point of it forms a hypocycloid. It is necessary to put $p = 2r(R + r)$; $a > R + 2r$ if the opening in the upper part of the surface is taken in the form of an epicycloid (5), Fig. 3a, and to put $p = -2r(R - r)$, $a > R$, if the opening in the upper part of the surface is taken in the form of a hypocycloid (6), Fig. 3b.

A surface of umbrella type with parabolic generatrixes and a round opening in the upper part has an epicycloid (5) in the foot contour z = 0 (Fig. 4a) or a hypocycloid (6), Fig. 4b. An arbitrary cross-section of an examined surface by a plane passing through the axis of the surface coinciding with a coordinate axis Oz will be a parabola. These parabolas can be described by Cartesian equation, $z = h - b(\varphi)(x - a)^2$, $0 \le z \le h$; $b(\varphi)$ is a variable parameter of the parabolas. The upper border of a surface is a circle with the radius a and with a center in the point with coordinates (0; 0; h). So, the picks of generating parabolas lie on the border circle.

Parametrical equations of the surface examined can be expressed as

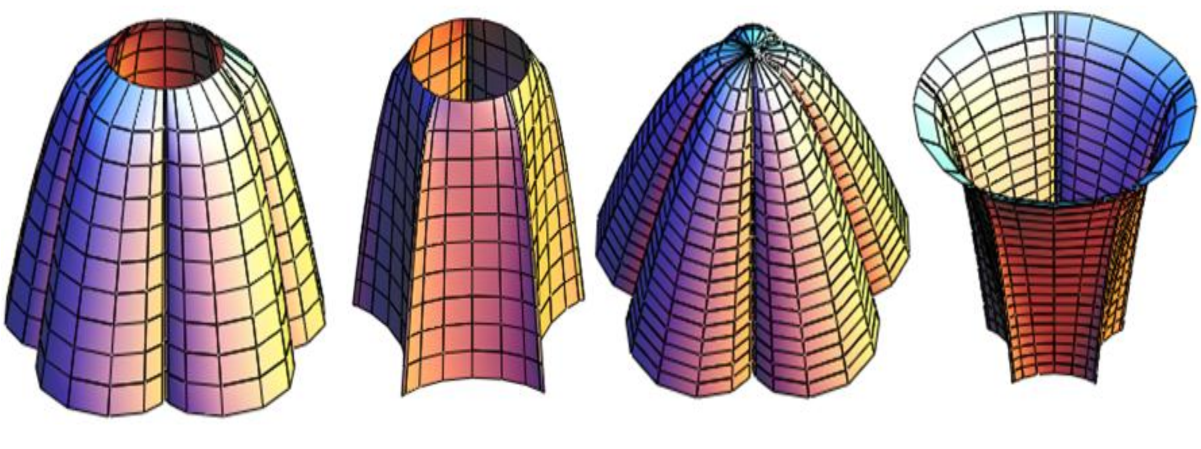
$$x = x(z,\varphi) = \left\{a + [v(\varphi) - a]\sqrt{\frac{h - z}{h}}\right\}\frac{x(\varphi)}{v(\varphi)},$$

$$y = y(z,\varphi) = \left\{a + [v(\varphi) - a]\sqrt{\frac{h - z}{h}}\right\}\frac{y(\varphi)}{v(\varphi)},$$

$$z = z,$$

where $v(\varphi) = \sqrt{R^2 + p(1 - \cos n\varphi)}$ is a polar radius of the hypocycloid or epicycloid lying in the foot of the surface but $v(\varphi) = \sqrt{x^2(\varphi) + y^2(\varphi)}$; $0 \le z \le h$; an angle φ is not a polar angle; $0 \le \varphi \le 2\pi$; h is the maximum height of the surface of umbrella type.

It is necessary to assume $p = 2r(R + r)$, $a < R$, if the foot was taken in the form of an epicycloid (Fig. 4a) or $p = -2r(R - r)$, $a < R - 2r$, if the foot has the form of a hypocycloid (Fig. 4b). Thus, in the cross sections of the considered surface by a plane z = 0, an epicycloid (6) or a hypocycloid (7) lies depending on a value of parameter p and coordinates x(φ) and y(φ).



(a)	(b)	(c)	(d)
a = 0,5 m; h = 2 m	a = 0,5 m; h = 2 m	a = 0,5 m; h = 2 m	a = 0,5 m; h = 2 m

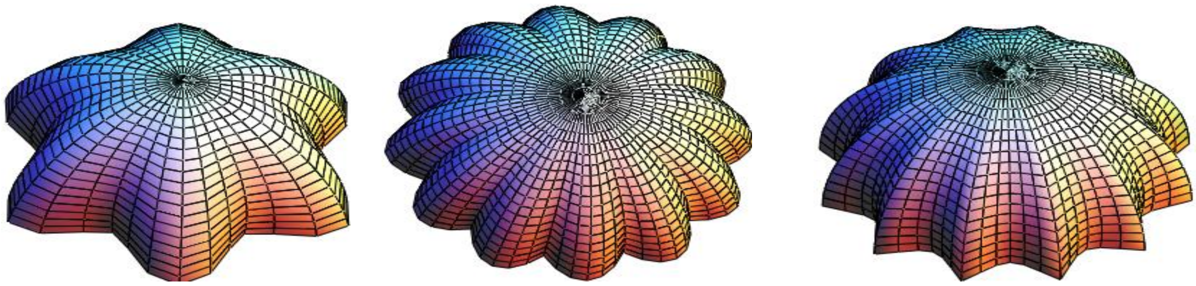
Fig. 4. A surface of umbrella type with parabolic generatrixes and a round opening in the upper part ($R = 1_M$, n = 5)

In the cross sections of the surface by a plane z = h, a border circle $x^2 + y^2 = a^2$ lies.

If one assumes a = 0 then the examined surface degenerates into a paraboloid of revolution with cycloid crimps (Fig. 4c). Having assumed $p = -2r(R - r)$, $a > R - 2r$, we shall derive a surface shown in Fig. 4d with a hypocycloid in the foot.

IV. UMBRELLA-FORM SURFACES WITH A CONTOUR SPHERICAL SURFACE

A crimped sphere (Fig. 5a) has a circular sinusoid (1) in the foot. A crimped sphere with external crimps (Fig. 5b) has a circular wavy curve (2) with $\delta = 1$ in the foot with picks directed from the center of the foot. A crimped sphere with inner crimps (Fig. 5c) has a circular wavy curve (2) with $\delta = -1$ in the foot with picks directed only inside the circular foot.



(a) (b) (c)

Fig. 5. Umbrella-form surfaces with a contour spherical surface: (a) a crimped sphere; (b) a crimped sphere with external crimps; (c) a crimped sphere with inner crimps

A crimped sphere (Fig. 5a) may be defined by the following parametrical equations:

$$x = x(\varphi,v) = [R\cos v + a(1 - \sin v)\cos n\varphi]\cos\varphi,$$
$$y = y(\varphi,v) = [R\cos v + a(1 - \sin v)\cos n\varphi]\sin\varphi,\ z = z(v) = R\sin v,$$

where v is an angle taken from the xOy plane in the direction of the Oz axis; $0 \le z \le R$; $0 \le \varphi \le 2\pi$; $0 \le v \le \pi$. The parametrical equations presented were derived with the help of a method described in the third section of the paper. In the cross-section of a crimped sphere by the plane z = const or, what is the same, assuming $v = v_o$ = const we shall obtain the circular sinusoids

$$x = x(\varphi) = [R\cos v_o + a(1 - \sin v_o)\cos n\varphi]\cos\varphi,$$
$$y = y(\varphi) = [R\cos v_o + a(1 - \sin v_o)\cos n\varphi]\sin\varphi,$$
$$z = R\sin v_o. \quad (7)$$

Parametrical equations of a crimped sphere with external crimps (Fig. 5b) can be represented as

$$x = x(\varphi,v) = [R\cos v + a(1 - \sin v)|\cos n\varphi|]\cos\varphi,$$
$$y = y(\varphi,v) = [R\cos v + a(1 - \sin v)|\cos n\varphi|]\sin\varphi,$$
$$z = z(v) = R\sin v. \quad (8)$$

where $0 \le z \le R$; $0 \le \varphi \le 2\pi$; $0 \le v \le \pi$, $\delta = 1$.

Parametrical equations of a crimped sphere with inner crimps (Fig. 5c) can be written by analogy with the equations (8) but with $\delta = -1$.

The crimped spheres presented in Fig. 5 have R = 1 m; a = 0,24 m; n = 6; $0 \le v \le \pi/2$.

Undoubted practical and theoretical interest can be aroused by spheres with cycloid crimps which are of two types. A sphere with external cycloid crimps (Fig. 6a) has an epicycloid (5) in the foot. A sphere with inner cycloid crimps (Fig. 6b) has a hypocycloid (6) in the foot with picks directed only inside the circular foot.

(a) n = 12 (b) n = 6

Fig. 6. Umbrella-form surfaces with a contoured spherical surface: (a) a sphere with external cycloid crimps; (b) a sphere with inner cycloid crimps

Parametrical equations of a sphere with external cycloid crimps (Fig. 6a) can be expressed as

$$x = x(u,\varphi) = [(R + \delta r)\cos\varphi - r\cos(n + \delta)\varphi]\cos u,$$
$$y = y(u,\varphi) = [(R + \delta r)\sin\varphi - r\sin(n + \delta)\varphi]\cos u,$$
$$z = z(u) = R\sin u, \qquad (9)$$

where $\delta = 1$, u is the angle taken from the xOy plane in the direction of the Oz axis; $0 \le z \le R$; $0 \le \varphi \le 2\pi$; $0 \le u \le \pi/2$. In the cross-section of an examined surface by the planes z = const or, what is the same, assuming $u = u_o$ = const, we shall have the epicycloids

$$x = x(\varphi) = [(R + r)\cos\varphi - r\cos(n + 1)\varphi]\cos u_0,$$
$$y = y(\varphi) = [(R + r)\sin\varphi - r\sin(n + 1)\varphi]\cos u_0,$$

with n = const.
Coefficients of fundamental forms of a surface (9) and its Gaussian curvature K can be written in the following forms:

$$A^2 = 2r(R + r)(1 - \cos n\varphi)\sin^2 u + R^2,$$
$$F = -R(R + r)\sin u\cos u\sin n\varphi,$$
$$B^2 = 2(R + r)^2(1 - \cos n\varphi)\cos^2 u,$$
$$L = \frac{Rr(R+r)(2+n)\cos u}{\sqrt{A^2B^2 - F^2}}(1-\cos n\varphi),\ M = 0,$$
$$N = \frac{R(R+r)^2(2+n)\cos^3 u}{\sqrt{A^2B^2 - F^2}}(1-\cos n\varphi),$$
$$K = \frac{R^2 r(R+r)^3(2+n)^2\cos^4 u}{(A^2B^2 - F^2)^2}(1-\cos n\varphi)^2 \ge 0. \qquad (10)$$

It follows from (10) that a surface (9) is mainly a surface of positive Gaussian curvature besides the plane lines of joints of identical elements of the surface where K = 0.
Parametrical equations of a sphere with inner cycloid crimps can be expressed by formulas (9) with $\delta = -1$.

The spheres with cycloid crimps presented in Fig. 6a and in Fig. 6b have R = 1 m; $0 \le v \le \pi/2$. A parameter n is shown in the corresponding figures.

V. . SURFACES OF UMBRELLA-TYPE FORMED BY SEMI-CUBICAL PARABOLAS

In certain cases, umbrella-type surfaces on a cycloid plan formed by semi-cubical parabolas can attract the attention of engineers and geometricians alike. These surfaces have an epicycloid (5) or a hypocycloid (6) in the foot. In every cross-section of an examined surface by a plane passing through the axis of the surface coinciding with a coordinate axis Oz, semi-cubic parabolas will lie. Cartesian equation of a family of semi-cubic parabolas is

$$x = a(\varphi)(h - z)^{3/2},$$

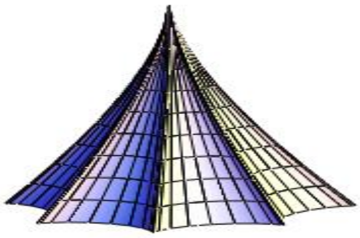
$0 \le z \le h$; $a(\varphi)$ is a variable parameter of the semi-cubic parabolas.
Parametrical equations of a surface of umbrella-type with an epicycloid in the foot (a surface with external crimps, Fig. 7a) are written as

$$x = x(u,\varphi) = u^{3/2}[(R + \delta r)\cos\varphi - r\cos(n + \delta)\varphi],$$
$$y = y(u,\varphi) = u^{3/2}[(R + \delta r)\cos\varphi - r\cos(n + \delta)\varphi],$$
$$z = z(u) = h(1 - u), \qquad (11)$$

where $\delta = 1$, $0 \le u \le 1$; u is a dimensionless parameter; $0 \le z \le h$; h is the maximum height of the surface (11); $0 \le \varphi \le 2\pi$. There is an epicycloid in every cross-section of a surface (13) by a plane u = const. The coordinate line u = 1 coincides with the foot epicycloids. The singular point is in the pick of the surface.

Parametrical equations of a surface of umbrella-type with a hypocycloid in the foot (a surface with inner crimps, Fig. 7b) we assume by analogy with the equations (11) but with $\delta = -1$, $n \neq 2.$ There is a hypocycloid in every cross-section of a surface (11) by a plane u = const. The singular point is in the pick of the surface (Fig. 7b).

(a)
n = 8; h = 2m; R = 1m



(b)
n = 8; h = 2m; R = 1m

Fig. 7. Surfaces of umbrella-type formed by semi-cubical parabolas: (a) a surface with external crimps; (b) a surface with inner crimps

VI. AN UMBRELLA-TYPE SURFACE WITH ASTROIDAL LINES OF LEVEL FORMED BY BIQUADRATIC PARABOLAS

Assume that an astroid x = $a\cos^3 t$, y = $a\sin^3 t$, lies in the xOy plane; then its polar radius will be $\rho = a\sqrt{\sin^6 t + \cos^6 t}$, where a parameter t is equal to an angle between the Ox axis and a straight line joining centers of a stationary circle with a radius R and rolling circle with radius r. A mobile circle rolls along the stationary circle inside of it. Any point of rolling circle with radius r forms an astroid; R/r = 4.

Parametrical equations of an examined surface were derived by the author:

$$x = x(t,u) = au^{1/4}\cos^3 t,\ y = y(t,u) = au^{1/4}\sin^3 t,$$
$$z = z(u) = H(1 - u), \qquad (12)$$

where u is a dimensionless parameter; $0 \le u \le 1$; $0 \le t \le 2\pi$; H is a maximum height of the surface, i.e., this is the distance from the surface foot to the highest point of the surface along the Oz axis. An examined surface of negative Gaussian curvature is presented in a non-orthogonal conjugate system of the curvilinear coordinates t, u. In the point u = 0 only, the surface has zero values of Gaussian and mean curvatures. Hence, the pick of the surface is a flat point.
The surface having a = 1 m; H = 2 m; $0{,}05 \le u \le 1$; $0 \le t \le 2\pi$ is shown in Fig. 8.

Fig. 8. An umbrella-type service with astroidal lines of level formed by biquadratic parabolas

For a surface given by the equations (12), a Cartesian equation was derived in the form:

$$z = \left\{1 - \left[\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3}\right]^6\right\}H.$$

An umbrella-type surface with astroidal lines of level formed by biquadratic parabolas can be given also by the following equations

$$x = x(r,\varphi) = r\cos\varphi,\ \ y = y(r,\varphi) = r\sin\varphi,$$
$$z = z(r,\varphi) = \left[1-\left(\frac{r}{a}\right)^4\left(\cos^{2/3}\varphi+\sin^{2/3}\varphi\right)^6\right]H.$$

VII. AN UMBRELLA SURFACE FROM THE PARTS OF TRANSLATIONAL SURFACE FORMED BY A GENERATING CIRCLE OF CONSTANT RADIUS

A joint-stock company «AQUA» (Design and Building) ordered a project of an entertainment center for possible building in the forestland in Moscow. As was assumed, the building must include two umbrella shells in itself. One shell of positive Gaussian curvature and another shell of negative Gaussian curvature were chosen. At first, the shells shown in Figs. 4a, b were offered for the application. Then it was decided to simplify a problem of middle surface formation and to assume the meridians of a contour spherical surface as contour curves of the shell elements, but a generating circle of constant radius was taken as a curve of another family of a translational surface with a round pole in the pick.

Assume a spherical surface with a radius a, given by the following equations:

$$x(u,\theta) = a\sin(u)\cos(\theta);\ y(u,\theta) = a\sin(u)\sin(\theta);\ z(u,\theta) = a\cos(u) \tag{13}$$

Let us set a curve $z_0 = a\cos(u_0)$ on a sphere parallel to the xOy plane. So, we have a circle with radius $R_0 = a\sin(u_0)$ in this cross-section. Let us also set a sector on the sphere limited by meridians $\theta = \pm\theta_0$.

Hence, a circle on the sphere in the cross-section z_0 is taken as the generatrix of the translational surface. The generating circle intersects two directing meridians $\theta = \pm\theta_0$ of the contour sphere (Fig. 9). Obviously, when a generating circle moves along the z axis, its center moves along the x axis. A cross-section of a translation surface is shown in Fig. 9b after removal of a generating circle along the z axis into the position z(u). In that case, a center of the generating circle moved along the x axis at the distance x = r(u). A position of a center of the generating circle is defined due to the condition of equality of the chords of the spherical element and a sector of the generating circle

$$R(u)\sin(\theta_0) = R_0\sin(V(u))\ \ \text{или}\ \ \sin(u)\sin(\theta_0) = \sin(u_0)\sin(V(u)).$$

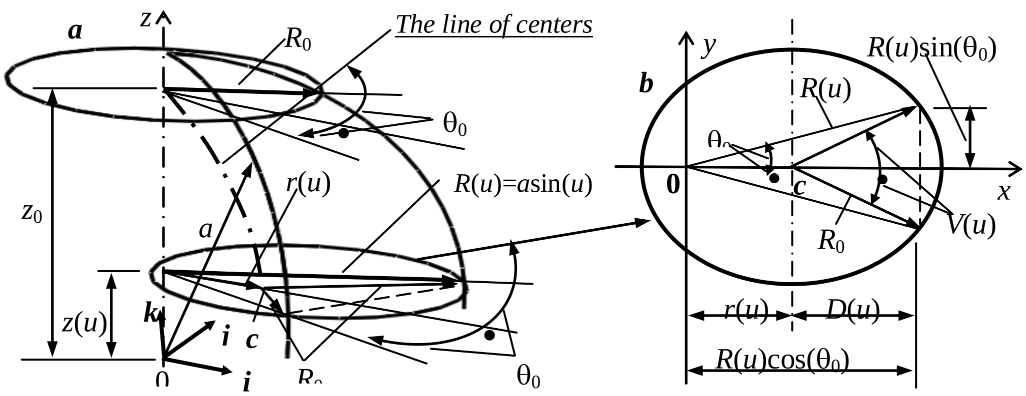
Hence,

$$V(u) = \arcsin\left[\frac{R(u)\sin(\theta_0)}{R_0}\right] = \arcsin\left[\frac{\sin(\theta_0)}{\sin(u_0)}\sin(u)\right]. \tag{14}$$

Studying Fig. 9b, V. N. Ivanov [7] derived

$$D(u) = \sqrt{R_0^2 - R^2(u)\sin^2(\theta_0)} = a\sqrt{\sin^2(u_0) - \sin(u)\sin^2(\theta_0)}\ , \tag{15}$$

$$r(u) = R(u)\cos(\theta_0) - D(u).$$



(a) (b)

Fig. 9. A translational surface formed by a generating circle of constant radius

From (15), we can see that $a\sin(u)\sin(\theta_0)$ must not be more a radius of the generating circle $R_0 = a\sin(u_0)$, so

$$u \le \arcsin(\sin(u_0)/\sin(\theta_0)).$$

It is clear from (16) and (14) that the ultimate value of a coordinate u satisfies the value of an angle parameter $V(u) = \pi/2$, and that is why a chord AB is equal to a diameter of the generating circle $2R_0$. A limitation (16) works if $\theta_0 < u_0$. An equator of the contour sphere determines a plane of symmetry of a translational surface. If $\theta_0 < u_0$ then one can have not joining symmetrical compartment of the surface in lower and upper parts of the contour sphere.

The line of centers $r_c(u)$ of the translational surface can be defined by the following vector equation

$$r_c(u) = r(u)i + z(u)k.$$

In that case, a vector equation of an examined translational surface with directing meridians of the contour sphere can be written as

$$\rho(u,\theta) = r_c(u) + R_0e(\theta), \tag{17}$$

where $e(\theta) = i\cos(\theta) + j\sin(\theta)$ is an equation of a circle of unit radius in the xOy plane. A translation surface intersecting a contour sphere is presented in Fig. 10a ($u_o = \pi/6$; $\theta_0 = \pi/8$; $u = \pi/6 \div \pi/2$). Contour meridians of the contour sphere do not coincide with coordinate lines of the translation surface. The equations of these coordinate curves are defined by the equations of the translation surface (17) after substitution θ = const in them.

Assume a system of curvilinear coordinates including contour meridians of the contour sphere in itself. Introduce a variable parameter

$$v = v(u,\theta) = p(u)\theta; \quad p(u) = V(u)/\theta_0 = \arcsin\left[\frac{\sin(\theta_0)}{\sin(u_0)}\sin(u)\right]/\theta_0$$

in the cross-section u = const.

A vector equation of the translational surface with curvilinear coordinates including contour meridians of the contour sphere can be expressed as

$$\rho(u,\theta)v = r_c(u) + R_0e(u,v)$$

where $e(u,v) = e[p(u)\theta] = i\cos[p(u)\theta] + j\sin[p(u)\theta]$. Changing an angle coordinate θ within the limits $-\theta_0 \le \theta \le \theta_0$ we can have a fragment of the translational surface limited by the directing meridians of the contour sphere (Fig. 10b). Set $\theta_0 = \pi/\kappa$ where κ is an integer. Adding surface fragments successively with the help of φ-angle turn of the initial fragment one can form a closed umbrella surface, $\varphi = 2\theta_0 i$ ($i = 1, 2,\ldots, \kappa - 1$). The umbrella surface with $u_0 = \pi/6$; $\theta_0 = \pi/6$ is shown in Fig. 10c.

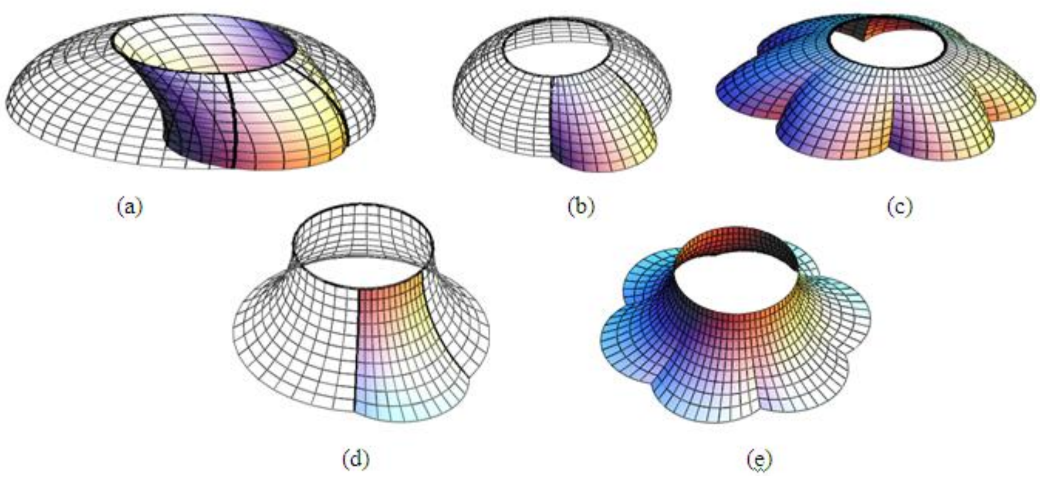


Fig. 10. Two types of umbrella surfaces from the parts of translational surface: (a) a translation surface intersecting a contour sphere; (b) a fragment of the translational surface of positive Gaussian curvature limited by the directing meridians of the contour sphere; (c) the umbrella surface of the first type; (d) a fragment of the translational surface of negative Gaussian curvature limited by the directing meridians of the contour torus; (e) the umbrella surface of the second type

Let us study the forming translational surface with a generating circle and directing meridians of a contour torus. Parametrical equations of a torus can be written as

$$x(u,\theta) = (b - a\sin(u))\cos(\theta);\ y(u,\theta) = (b - a\sin(u))\sin\theta);\ z(u,\theta) = a\cos(u),$$

where a is the radius of a generating circle of the torus; b is the distance the axis of revolution from the center of the generating circle.

Assume a fragment of the torus formed by the lower concave part of a circle with $u = \pi/2 \div \pi$ as a contour surface of a translational surface which is a surface of negative Gaussian curvature. Taking $u = u_0$ we can have a generating circle with the radius $R_0 = b - a\sin(u_0)$.

By analogy with the contour sphere, one can form a translational surface with a generating circle and directing meridians of the contour torus taking $\pm\theta_0$ and removing a generating circle with radius R_0 along directing meridians of the torus. Analyzing translational surface formations, we can note the identity of all formulas. The difference is only in the way of the writing of a formula $R(u) = a\sin(u)$ for the contour sphere and a formula $R(u) = b - a\sin(u)$ for the contour torus.

A fragment of the translational surface with the generating circle of constant radius and directing meridians on the contour torus is shown in Fig. 10d. In Fig. 10e, the umbrella surface is presented with

$$a = 1\ m;\ b = 1{,}5\ m;\ u_0 = \pi/2;\ \theta_0 = \pi/6;\ u = (\pi/2 \div \pi).$$

VIII. RESULTS

In this paper, parametrical equations of the 16 new umbrella-type surfaces, derived by the author, and the umbrella surface from the parts of translational surface formed by a generating circle of constant radius are presented and the influence of constants containing in the equations of these surfaces on their form are studied. Giving appointed interval of changing of a geometric parameter, one can see a picture of the change of a form of the surface in the multimedia regime. Visualization of the surfaces of umbrella type (Figs. 1–10) was realized with the help of the computer program MathCAD. The rest of surfaces of umbrella type known today are presented in the encyclopedia [8]. Some of H. Isler's shells [9] may be added to umbrella shells.

IX. CONCLUSIONS

Revival of the highly artistic umbrella form is possible today only in the shape of thin-walled structures made of modern high-strength materials. The presence of a wide choice of umbrella surfaces and surfaces of umbrella type gives an opportunity for engineers and architects to choose a form which will satisfy the working [6], architectural, aesthetic, and strength [1, 10] requirements. One must not forget requirements of midi ergonomics. At present, umbrella surfaces and surfaces of umbrella type, apart from the surfaces pointed in General Information, are used in radio engineering, light industry, modeling of surfaces of biological forms, and design of quickly erecting buildings. Additional information on the application of umbrella shells can be taken from [11]. The authors hope that umbrella and umbrella-type shells will be widely used in different areas of human activity. Having equations of middle surfaces of thin-walled structures one can start computer modeling of shell buildings or strength analysis of them [1, 10–14], and it will promote wider introduction of them in real practice.

REFERENCES

[1]. B. Peerdeman, Analysis of thin concrete shells revisited: Opportunities due to innovations in materials and analysis methods, master's thesis, Delft University of Technology, the Netherlands, 2008.

[2]. H. J. Cowan, P. R. Smith, and W. K. Chow, Dictionary of architectural and building technology (London, UK: Taylor & Francis, 2004).

[3]. P. Draper, M. E. M. Garlock, and D. P. Billington, Structural optimization of Félix Candela's hypar umbrella shells, Journal of the International Association for Shells and Spatial Structures, 51(1), 2012, 59–66.

[4]. D. F. Tully, Inverted, doubly-curved umbrella, hyperbolic paraboloid shells with structurally integrated upper diaphragm, Patent 4137679, February 6, 1979.

[5]. G. Brankov, Corrugated shell structures, Bulg. Academy of Science, 1961.

[6]. M. H. Imam, Shape optimization of umbrella-shaped concrete shells subjected to self-weight as the dominant load, Computer & Structures, 69(4), 1998, 513–524.

[7]. V. N. Ivanov, and S. N. Krivoshapko, Design of umbrella surfaces from the parts of translational surfaces formed by the circle of constant radius. Structural Mechanics of Engineering Construction and Buildings, 1, 2011, 3–10.
[8]. S. N. Krivoshapko, and V. N. Ivanov, Encyclopedia of analytical surfaces [In Russian] (Moscow: LIBROCOM, 2010).
[9]. T. Kotnik, and J. Schwartz, The architecture of Heinz Isler, Journal of the International Association for Shells and Spatial Structures, 52(3), 2011, 185–190.
[10]. J. J. del Coz Diaz, L. Panalver Lamarca, P. J. Garcia Nieto, A. L. Martinez-Luengas, and J. A. Vilán Vilán, Non-linear analysis of a reinforced concrete sheet cover (umbrella) of 40 m diameter by the finite element method, Applied Mathematics and Computation, 184(1), 2007, 37–51.
[11]. S. N. Krivoshapko, I. A. Mamieva, Umbrella surfaces and umbrella-type surfaces in the architecture, Prom. i grazhd. stroitelstvo, 7(1), 2011, 27–31.
[12]. N. Abbushi, and A. Yu, Wavy domes, Structural Mechanics of Engineering Construction and Buildings, 11, 2002, 49–58.
[13]. V. A. Lebedev, Thin-walled umbrella shells [in Russian], GSI, 1958.
[14]. L. Hollaway, and B. Bell, An analytical and experimental technique for testing GRP domes, Composite, 9(2), 1978, 105–111.